Evolution and formation of dispersive-dissipative patterns

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A variety of interfacial phenomena, including non-Boussinesq and Marangoni effects are described by a dispersive-dissipative model $u_\tau + \alpha u u_\xi + \beta u_{\xi\xi\xi} + [(1+2u)u_\xi]_\xi + \gamma u_{\xi\xi\xi\xi} = 0$. A critical surface $\alpha = \alpha_c(\beta,\gamma)$ is found such that for $\alpha < \alpha_c(\beta,\gamma)$ the amplitude becomes unbounded within a finite time and the model breaks down. For $\alpha > \alpha_c(\beta,\gamma)$, if the initial perturbation is not too large, bounded patterns emerge. The interaction between dispersion and advection dislocates the critical surface (favorably when dispersion and convection cooperate) and suppresses the temporally irregular nature of the resulting patterns. In the first of the two regularized variants of the model considered, the amplitude runaway is mitigated and a formation of cusps is observed. In the second variant with a quadratic dispersion, the emerging solutions are bounded save for a strip in a parameter space, where both the amplitude and the gradients were found to grow at competing rates. [S1063-651X(97)51202-5]

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The amazing success of mathematical models to describe the physical reality tends to obliterate their approximate nature, and thus the tentative nature of the information gathered via the analysis of these models. When nonlinearity is concerned, the conventional derivation based on asymptotic ordering may not suffice to render a well-balanced model. Unlike a linear theory, where a direct relation between cause and effect can be estimated *a priori*, in nonlinear theory a seemingly minor constituent may cascade the dynamics into a catastrophy. There are many examples in the recent scientific literature where a seemingly slight generalization of a model renders far worse results to the point of ill-posedness. The generic model presented in this Communication describes a formation of patterns due to an interaction between advective, dispersive, and dissipative forces. In this model the competition between long-wave destabilizing and short-wave stabilizing forces runs out of balance in a certain parameter domain, causing an amplitude runaway within a finite time and a collapse of the model. We delineate the domain(s) of well-posedness of this model as a physico-mathematical entity, and then proceed to present both mathematical and physical strategies for its regularization to render a well-balanced model. In particular, we shall show that when the model equation describes the evolution of an interface separating two liquids in a horizontal cylinder, the regularization enables us to follow the evolution of the interface toward rupture and a formation of drops.

A poor description of the ultraviolet regime by most macroscopic models is a direct consequence of the way these models are derived, namely, under the assumption of both small amplitudes and small gradients. Though the derivation is consistent with the intention that these models be used within these confines, it is the nature of nonlinear processes to induce, oblivious of their desirability, all scales. Consequently the large amplitude and/or large gradient behavior of these models has little to do with their original counterparts. The ultraviolet falsetto may be a harmless annoyance; however, at other times an artificial catastrophy in amplitude or gradient results. Another aspect of the ultraviolet mismatch is the occasional elimination by the conventional expansion of a natural critical threshold(s), such as the onset of rupture of the interface or an overturn of waves, yielding instead an overly calm model equations [such as the Korteweg–de Vries (KdV) or the Kuramoto-Sivashinsky (KS) equation]. Neither the KS nor the KdV contain any information regarding a critical onset. In fact, these models always predict smooth patterns. To unfold such an onset one has to find and restore the relevant mechanism(s) dispensed with in a conventional expansion.

Returning to our model equation we write

$$f_t + a f f_x + b f_{xxx} + c f_{xx} + d(f^2)_{xx} + \sigma f_{xxxx} = 0, \qquad (1)$$

where a,b,c,d,σ are constants and $0 \leqslant x \leqslant L$. The model is a natural extension, appended with dispersion, of the KS model, but also with the destabilizing backward diffusion being quadratic. This extension has a dramatic impact on the resulting dynamics, because unlike the KS, the quadratic instability may overcome the short-wavelength stabilization causing an amplitude runaway, unless stabilized by other mechanisms. As to the physical origins of this model, we note the Benard-Marangoni convection in a shallow layer with the free deformable boundary heated from the air side [1,2] and the Eckhaus instability of waves [3]. The solidification of dilute binary alloys [4] has also been found to be described by an equation similar to Eq. (1) with $a=b=0$ or with $a=0, b \neq 0$. A model of bioconvection and the effect of fluid's compressibility also lead to a similar equation [5]. When $a=b=0$ any initial perturbation either evolves into a

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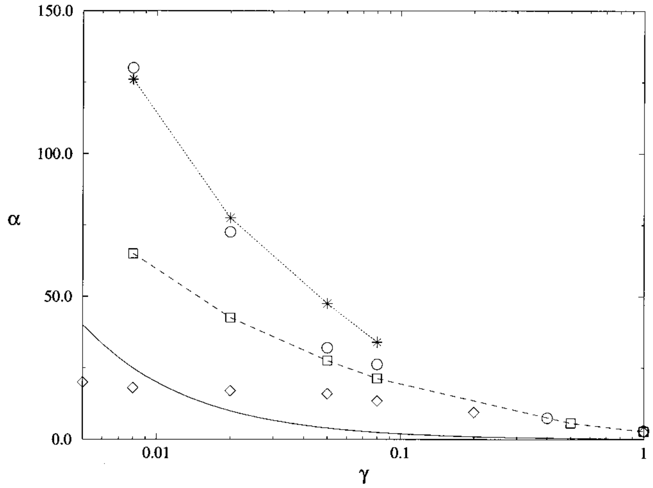

FIG. 1. Data points for the critical surface $\alpha=\alpha_c(\beta,\gamma)$ for three values of β: $\diamond-\beta=-0.1;\square-\beta=0;\bigcirc-\beta=0.1$. All $\gamma>1$ states are linearly stable. When $\alpha>\alpha_c(\beta,\gamma)$, the solutions of Eq. (2) evolving from small-amplitude initial data are bounded, while for $\alpha<\alpha_c(\beta,\gamma)$ they runaway. The solid $\alpha=2\beta\gamma^{-1}$ curve represents soliton solutions, Eq. (6), for $\beta=0.1$. The * points represent the critical line of Eq. (7) (the C-KS) with $\beta=0$.

spatially uniform state or, if linearly unstable, causes an amplitude blow-up in a finite time [4,6].

In a rescaled form Eq. (1) reads

$$u_\tau+\alpha u u_\xi+\beta u_{\xi\xi\xi}+[(1+2u)u_\xi]_\xi+\gamma u_{\xi\xi\xi\xi}=0, \qquad (2)$$

where $f=cu/d, x=L\xi/(2\pi), t=L^2\tau/(4\pi^2 c)$ with $\alpha=aL/(2\pi d), \beta=2\pi b/(cL), \gamma=4\sigma\pi^2/(cL^2)$. We study Eq. (2) on a periodic domain $0\leqslant\xi\leqslant 2\pi$, and for the initial data we assume $u(\xi,\tau=0)=(C)\mathrm{rand}(\xi)$, where $\mathrm{rand}(\xi)$ is a randomly distributed function in the range $(-1,1)$ and C is a small constant, $10^{-4}\leqslant C\leqslant 10^{-2}$.

The symmetry of Eq. (2) under $(u,\xi,\alpha,\beta)\to(u,-\xi,-\alpha,-\beta)$ enables one to study Eq. (2) for $\alpha>0$ with $-\infty<\beta<+\infty$.

Stability. Our main results are displayed in Fig. 1. To solve numerically Eq. (2) we have used a time-splitting method. The three sets of data points, represented by $\diamond$, $\square$, and $\bigcirc$, were obtained for $\beta=-0.1$, 0, and 0.1, respectively. Each has the following property: for any (α,γ) pair located under the corresponding critical curve the solution of Eq. (2) explodes in a finite time, while for those located above, for a small initial data, a bounded solution emerges. This defines the critical function $\alpha=\alpha_c(\beta,\gamma)$ that separates between two very different evolution paths. Since this demarcation is done numerically, it is possible that the curve thus traced is actually a narrow strip with its own character. In this kind of affair a rigorous mathematical analysis is needed (c.f. Ref. [7]). Numerical calculations reveal that the domain of amplitude runaway broadens with the increase of β and shrinks with its decrease. Observe also that the domain of blow-up narrows with the increase of γ and terminates at $\gamma=1$. For $\gamma>1$ the trivial state is linearly stable. The amplitude saturation results when an enhanced energy transfer from low-k modes into the higher modes (via the nonlinear advection $\alpha u u_\xi$), leads to an effective short-wavelength-energy dissipation via the $\gamma u_{\xi\xi\xi\xi}$ part. The increase in the advection coefficient α enhances this process and lowers the saturation amplitude. A similar effect was noted for the Benney equation [8].

It is important to stress that the emerging patterns in the stable domain emerged out of small initial perturbations. Nevertheless, certain large initial perturbations were found to run away, which implies that even the stable patterns may be destroyed if strongly perturbed. We do not know yet how to characterize these perturbations apart from clear evidence that their measure is not in the maximum norm.

To understand the dispersive-advective interaction take $u\sim\exp(ik\xi+i\omega\tau)$ (k and ω being the wave number and the wave frequency, respectively). Then for the linearized $u_\tau+\beta u_{\xi\xi\xi}=0$ equation, $\omega=\beta k^3$. Thus, the phase velocity is positive (negative) for $\beta>0(<0)$ and the corresponding wave propagates to the left (right) in ξ. In the nonlinear problem this phase shift is coupled to the nonlinear advection, which causes both deformation and a displacement of the wave. We can formally write this interaction in a mixed notation as $(\alpha u-\beta k^2)u_x$. In a bounded domain its size sets a lower bound on k and thus enhances (hinders) the effective advection if $\beta<0$ $(\beta>0)$.

The marginally stable state that occurs at $\gamma=1$ enables an analytical glimpse into the problem. Exploiting the slow linear growth in the marginal vicinity, we expand $u=\Sigma_{n=1}^{\infty}\delta^n u_n(\overline{\tau},z)$ in terms of a small parameter δ, where $\gamma=1-\overline{\gamma}\delta^2, \overline{\gamma}>0$ and $\overline{\tau}=\delta^2\tau$, $z=\xi+\beta\tau$. Using Eq. (2) the functions u_i are calculated up to two first orders in δ: $u_1(\overline{\tau},z)=A(\overline{\tau})e^{iz}+$c.c, where A is a complex function yet to be determined. In the third order of expansion in δ calculations yield an amplitude equation in the familiar form

$$\frac{d|A|}{d\overline{\tau}}=\overline{\gamma}|A|-\kappa|A|^3, \quad \kappa=\frac{\alpha^2-8-3\alpha\beta}{3(4+\beta^2)}, \qquad (3)$$

where $|A|$ is the value of the complex function A and κ is the Landau constant. The bifurcation is therefore supercritical (subcritical) if $\kappa>0(\kappa<0)$.

When $\beta=0$, the bifurcation is supercritical ($\kappa>0$ and the solution is bounded) if $|\alpha|>\sqrt{8}$ and subcritical otherwise. This is in excellent agreement with our numerical studies, shown in Fig. 1. When $\beta\neq 0$, the bifurcation is supercritical when

$$\beta\langle\frac{\alpha^2-8}{3\alpha}, \quad \text{alternatively} \quad \alpha\rangle\frac{3\beta+\sqrt{9\beta^2+32}}{2}, \qquad (4)$$

where $\alpha>0$. The linear dispersion depends on its coupling to the advection to change the threshold for a blow-up. Without advection $\alpha=0$ (c.f. the solidification model), irrespective of dispersion κ is always negative and the bifurcation is always subcritical.

Patterns. Within the class of bounded solutions we distinguish between (a) temporally irregular solutions and (b) spatially ordered wavetrain solutions consisting of equal-amplitude humps. Similarly to the solutions of the KS, when $\beta=0$ the bounded solutions of Eq. (2) are irregular in a domain which supports a large number of linearly unstable

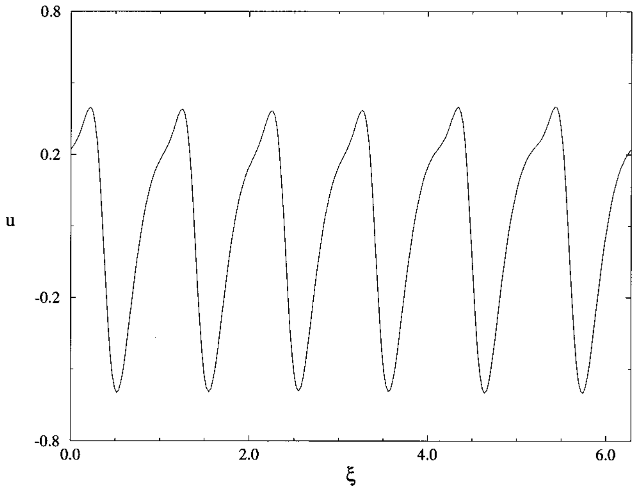

FIG. 2. Traveling-wave solution of Eq. (2) for $\alpha=50$, $\beta=-0.1$, $\gamma=0.008$ at $\tau=4$.

modes (small values of γ). As a measure of irregularity we use the ''energy'' E of the solution $u(\xi,t)$, $E=\int_0^L u^2(\xi,\tau)d\xi$.

Numerical studies reveal that in the presence of dispersion, $\beta\neq 0$, the irregular solutions are limited to a strip located above the critical curve in the $\alpha-\gamma$ plane. The spatially ordered solutions are located above this strip, which is to say that a higher α is needed to get regular patterns for the same values of β and γ. With an increase of γ ($\gamma<1$) the strip of irregular solutions shrinks and eventually disappears. Given an irregular pattern in order to regularize it, one must, at fixed α and γ, either lower the value of β or increase the value of α, for fixed values of β and γ. Alternatively, the same can be achieved by increasing γ (this corresponds to an increase in the surface tension) while freezing the other two parameters.

These properties are illustrated in Figs. 2 and 3 for $\gamma=0.008$ and $\beta=-0.1$. Figure 2 displays a spatially ordered traveling-wave solution for $\alpha=50$. The solution's energy, E, approaches in time a constant value (not shown) when the pattern is attained. Figure 3(a) shows a temporally irregular solution for $\alpha=30$, while Fig. 3(b) presents its energy E as a function of time τ.

The presence of dispersion induces both purely periodic waves and spatially damped aperiodic waves. In particular we find solitary waves. To this end Eq. (2) is rewritten first as

$$u_\tau-\beta\gamma^{-1}u_\xi+(\beta\gamma^{-1}+\partial_\xi)\partial_\xi(\gamma u_{\xi\xi}+u^2+u)=0, \quad (5)$$

which is possible if $\alpha=2\beta\gamma^{-1}$. In a moving frame of reference $\zeta=\xi+\beta\gamma^{-1}\tau$, which absorbs the linear advection, one has a family of stationary solutions given in terms of elliptic functions via $\gamma u_{\zeta\zeta}+u^2+u=\text{const}$. The solitary waves are given as

$$u(\zeta)=-(1+4\gamma s^2)/2+6\gamma s^2\cosh^{-2}(s\zeta), \quad (6)$$

where s is an arbitrary constant.

The demarcation line between the bounded and exploding solutions also helps to determine the stability of a pattern. In

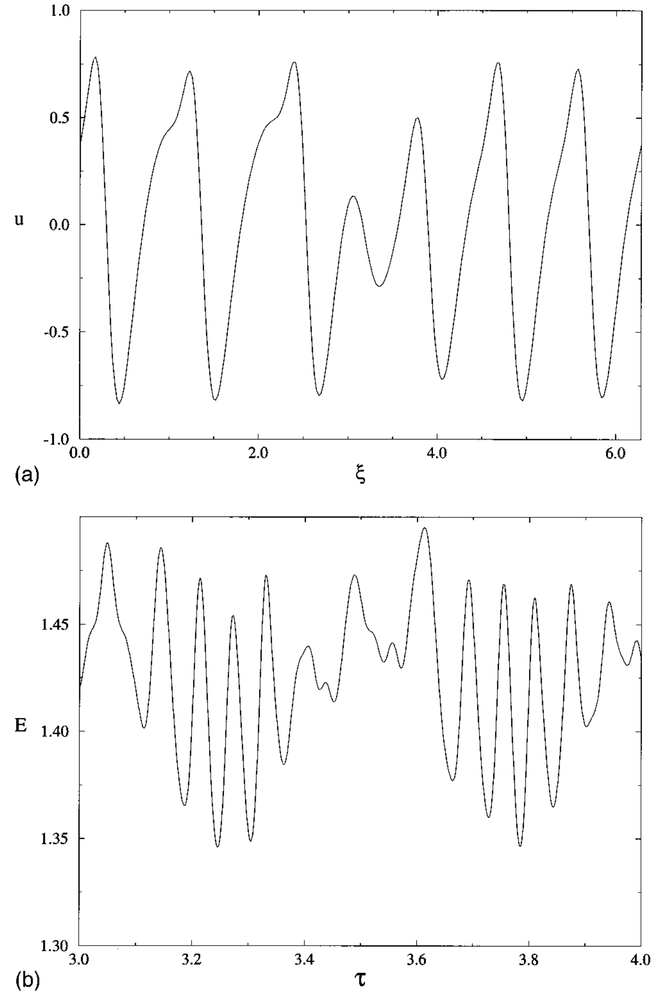

FIG. 3. (a) The irregular solution of Eq. (2) for $\alpha=30$, $\beta=-0.1$, $\gamma=0.008$ at $\tau=4$. (b) Its energy function E as a function of time τ.

the case of solitons (or periodic waves) we draw their solution line $\alpha=2\beta\gamma^{-1}$ on the $\alpha-\gamma$ plane (see Fig. 1) to find that it is located well within the blow-up domain. Therefore, these particular solitons cannot materialize. In fact, using some of these solitary solutions (for variety of s and $\gamma<1$) as an initial condition, results either in a decay into the trivial, spatially uniform solution or in a blow-up. The fate of solitons in the linearly stable $\gamma>1$ domain is still unclear.

A more general solution family is obtained if Eq. (5) is rewritten as

$$u_\tau-hu_\xi+\left(\frac{\alpha}{2}+\partial_\xi\right)\partial_\xi(\gamma u_{\xi\xi}+lu_\xi+u^2+2ru)=0, \quad (5')$$

where $h=r\alpha$, $l=\beta-\gamma\alpha/2$, and $8r=\gamma\alpha^2-2\alpha\beta+4$. Again, stationary solutions in a moving frame are available and are given via $\gamma u_{\zeta\zeta}+lu_\zeta+u^2+2ru=\text{const}$. Depending on the choice of constants the profile connecting upstream with the downstream may be monotone or aperiodic damped oscillations. The freedom to choose α enables us for a given β to choose a supercritical value (say, for $\beta=0$ and $\gamma=1$ take $\alpha=3$), and thus a stable pattern. The qualitative nature of these patterns is easily determined via their phase-plane analysis but, apart of exceptional cases, an explicit analytical forms for these waves are unavailable.

Regularization. Consider now

$$u_t+\alpha u u_\xi+\beta u_{\xi\xi\xi}+\left(\frac{u}{1-u}\right)_{\xi\xi}+\gamma u_{\xi\xi\xi\xi}=0. \qquad (7)$$

It is easily seen that Eq. (2) can be formally derived from Eq. (7) when its fractional term, according to the dictum of conventional asymptotics, is expanded in small u and quadratic terms are kept. Equation (7) (the C-KS) was recently shown [9] to describe the evolution of an unstable interface separating two liquids (oil and water) in a horizontal circular pipe. In fact the new term describes the destabilizing mechanism due to the low order part of the interfacial curvature in cylindrical symmetry. Numerical studies of Eq. (7) [9] reveal the existence of a critical threshold: for relatively large values of α the solutions for Eq. (7) evolving from an arbitrary initial data are bounded and smooth. However, for relatively small values of α the denominator in Eq. (7) vanishes and a singularity of the solution emerges in the form of a cusp having a finite amplitude. In this context these solutions describe a rupturing interface followed by the formation of experimentally observed [10] bubbles. Equation (7) may then be viewed as an effective, physically motivated, regularization of our prototype problem. Note that upon expansion of the fraction the regularizing effect spreads over all orders and will not be recovered if the expansion of the *overall problem* is carried up one more order. In fact, such an expansion yields a sixth-order partial-differential equation that is ill-posed due to a ''wrong'' sign of the highest order term. Needless to say, however, a different physical setting may call for a different regularization.

The same auxiliary conditions were applied to Eq. (7) as to Eq. (2). Using (*) as data points, we outline in Fig. 1 the critical surface, which is now of different kind, of Eq. (7). Under this curve the solutions form cusp(s), while above it they are bounded and appear to be smooth (c.f. Ref. [9]). It is important that the new critical curve is located above the original curve generated by Eq. (2). Note that in the domain between these curves both Eqs. (2) and (7) predict bounded, but very different, patterns. Of course, both cannot describe the same phenomenon; thus, if Eq. (2) is viewed as a simplified model of Eq. (7), its viability to correctly describe the physics comes to its end near the critical curve of Eq. (7) and not on its own critical curve of well-posedness. For parameters below the critical curve of Eq. (7), solutions of Eq. (2), independently of whether they explode or not, do not represent the physical problem. The generic nature of Eq. (2) implies its emergence in different setting then, say the one just presented. This in turn will call for a different regularization that may or may not have its own critical curve, which in turn will invalidate certain predictions provided by Eq. (2).

Finally, consider another regularization of Eq. (2). Here we argue that once an argument in favor of use of a quadratic backward diffusion has been made, at the same level of approximation one has to seek other quadratic terms. Motivated by a number of physical examples (c.f. Ref. [11]) we replace the linear dispersion in Eq. (2) with a quadratic dispersion $\mu(u^2)_{\xi\xi\xi}$. The combined effects of linear and quadratic dispersion will be presented elsewhere. Bifurcation analysis carried for the new equation near the critical $\gamma=1$ point assures supercritical bifurcation if $\mu>(5\alpha+\sqrt{9\alpha^2+128})/16$, or $\mu<(5\alpha-\sqrt{9\alpha^2+128})/16$ and boundedness of the solutions in full accord with our numerics. Thus, for every α there is a broad range of μ capable of stabilizing the flow. In particular, without advection, bifurcation is supercritical if $|\mu|>1/\sqrt{2}$. Unlike the previous model where critical curve defined half-space, here the quadratic dispersion limits the excluded zone to a strip. Our numerical studies reveal that in this strip the growth of the amplitude is softened and its growth competes with the growth rate of the gradients. So far, numerical studies have not yielded a convincing argument as to the outcome of this competition. This is left for future studies.

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